

A new type fit for the CKM matrix elements

Petre Dîţă*

Institute of Physics and Nuclear Engineering, P.O. Box MG6, Bucharest, Romania

(Dated: February 1, 2008)

The aim of the paper is to propose a new type of fits in terms of invariant quantities for finding the entries of the CKM matrix from the quark sector, by using the mathematical solution to the reconstruction problem of 3×3 unitary matrices from experimental data, recently found. The necessity of this type of fit comes from the compatibility conditions between the data and the theoretical model formalised by the CKM matrix, which imply many strong nonlinear conditions on moduli which all have to be satisfied in order to obtain a unitary matrix.

I. INTRODUCTION

In the description of the electroweak interactions within the Standard Model a fundamental rôle is played by the unitary CKM matrix, parametrised by four independent parameters: three mixing angles, θ_{ij} , and a CP violating phase, δ . On the other hand from experiments one finds numerical values for other quantities such as the moduli, or the angles of so called unitarity triangles, and by consequence the reconstruction of unitary matrices from experimental data is a central problem in the electroweak interactions. Only recently a mathematical solution to this problem was found, [1], and one aim of the paper is to explore its implications upon the electroweak phenomenology. One conclusion of the above cited paper was that it is necessary to provide phenomenological models for analysing data in terms of rephasing invariant quantities, i.e. the CKM matrix moduli, the Jarlskog invariant, J , [2], the area of any unitarity triangle, [3], or the phases of its complex entries [4]. In this paper we will choose the moduli as independent parameters because with this choice the reconstruction of a unitary matrix from data is essentially unique, see Ref. [1]. In the same time this choice allows us to define phenomenological models which lead to sets of non linear constraints which have to be satisfied by data in order to come from, or to be compatible with a unitary matrix. Hence the data could be compatible with unitarity, or even could disprove the CKM unitary model, i.e. nobody guarantees us that the moduli $|U_{ij}|$, the mixing angles θ_{ij} , or the phase δ extracted from experiments come from a unitary matrix. In phenomenological analyses it is usually assumed that irrespective how the measured data are, they are compatible to the existence of a unitary matrix, and in literature one finds statements such as: the most stringent test of CKM unitarity is the relation

$$V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 1 \quad (1)$$

see, e.g., Ref. [5], where V_{ij} are the measured moduli values, statement which is not true, because there is a natural embedding of unitary matrices into a larger class of

matrices, that of double stochastic ones, [6], and we have to find methods to discriminate between the two sets.

A 3×3 matrix M is said to be double stochastic if its elements satisfy the relations

$$m_{ij} \geq 0, \quad \sum_{i=1}^3 m_{ij} = 1, \quad \sum_{j=1}^3 m_{ij} = 1 \quad (2)$$

The unistochastic matrices, [6], are a subset of the double stochastic ones defined by

$$m_{ij} = |U_{ij}|^2 \quad (3)$$

where U is a 3×3 unitary matrix. Hence even if all the six relations such as (1) are exactly satisfied the corresponding matrix could be non-unitary.

The double stochastic matrices have an important property, they are a convex set, i.e., if M_1 and M_2 are double stochastic, so is their convex combination $\alpha M_1 + (1 - \alpha)M_2$, $\alpha \in [0, 1]$, as it is easily checked. This property is very important because it was the necessary ingredient to devise a method for doing statistics on unitary matrices.

The relations (2) together with the embedding relations (3) define a phenomenological model which allows to find formulae for the mixing angles $s_{ij} = \sin \theta_{ij}$ and $\cos \delta$ in terms of four independent moduli V_{ij} , and the compatibility condition between data and unitarity property leads to the most constraining unitarity condition, namely, $\cos \delta$ as function of moduli should take physical values, i.e. $-1 \leq \cos \delta \leq 1$, see Ref. [1].

The choice of four moduli as independent parameters is very appealing from a theoretical point of view, since, as we said before, in this case the reconstructed unitary matrix is essentially unique. On the other hand it is naturally to assume that the physical results of any phenomenological analysis of experimental data must be invariant with respect to the choice of the unitary matrix form, e.g., Kobayashi-Maskawa, [7], or Chau-Keung, [8], form. Although both the above cited parameterisations depend on mixing angles, θ_{ij} and CP non-conserving phase δ , only the last parameter is invariant, the numerical values for θ_{ij} depend on the chosen form. The mixing angles being not invariant quantities, it is better to avoid their use in phenomenological analyses. If the physical results are not invariant, this could be a signal that the

*Electronic address: dita@zeus.theory.nipne.ro

unitary model is not compatible with experimental data, or that our phenomenological data analysis is done with wrong parameters.

The above considerations imply that for defining a *novel type of CKM fits* we have to make clear a few natural assumptions, which we will state as axioms. Concerning the experimental side an obvious axiom could be the following:

1) *the numerical values for all the measured moduli, $|U_{ij}|$, should be the same irrespective of the physical processes used to determine them.*

The second axiom comes from the constraints imposed by data on the explicit theoretical tool used for doing phenomenological analyses. Thus the second axiom could be:

2) *the physical results obtained from data analyses must be invariant with respect to the choice of four independent*

invariant quantities used to parametrise the data.

Because in this paper we will use moduli as invariant parameters, and the number of four independent moduli groups equals 58, one gets 165 different forms for $\cos \delta$, which when computed by using real data one finds different values, instead of a single one. Thus the above axiom will have in this paper the form:

2a) *the physical results must be invariant with respect to all the choices of four independent moduli groups used to parametrise the data.*

Of course for defining phenomenological models and for a full reconstruction of unitary matrices from error affected data we have to use one explicit form of the CKM matrix, and in the following we use the form from Ref. [8], which, by making full use of its invariance at multiplication at left and at right with diagonal phases matrices, we write it as

$$U = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12}e^{i\delta} - c_{12}s_{23}s_{13} & c_{12}c_{23}e^{i\delta} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23}e^{i\delta} - c_{12}c_{23}s_{13} & -c_{12}s_{23}e^{i\delta} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix} \quad (4)$$

with the standard notation: $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$; θ_{ij} , $ij = 12, 13, 23$, denote the mixing angles and δ is the phase that encodes the CP -violation in the electroweak sector.

The paper content is as follows. In Section 2 we define two phenomenological models that will provide the necessary and sufficient conditions the data have to satisfy in order that they should come from a unitary matrix. Eventually by finding that data come from a unistochastic matrix we have to provide an algorithm for the reconstruction of U from the error affected data. In Section 3 we test the moduli data available from experiments, lattice computations, and global fits. We show that the experimental errors are quite large so there exists a continuum of unitary matrices compatible with data. The paper ends by Conclusion.

II. PHENOMENOLOGICAL MODELS

By phenomenological model we understand a relationship between the entries of a unitary matrix and the measured quantities, such as the moduli. Hence in the following we assume the knowledge from experiment of the moduli of unitary matrix entries such as (4), which we write as the entries of a positive matrix

$$V = \begin{pmatrix} V_{ud}^2 & V_{us}^2 & V_{ub}^2 \\ V_{cd}^2 & V_{cs}^2 & V_{cb}^2 \\ V_{td}^2 & V_{ts}^2 & V_{tb}^2 \end{pmatrix} \quad (5)$$

For the current state of the art concerning the determination of the above quantities see, e.g., Refs. [5] and [9]. As the notation suggests we make a clear distinction between the elements of the unitary CKM matrix U and the *positive* entries matrix V provided by data. In other words we make a distinction between the theoretical quantities U_{ij} and the experimental moduli V_{ij} , although in an ideal situation the relation $|U_{ij}| = V_{ij}$ will hold.

The main phenomenological problem is to see in what conditions from a matrix such as (5) one can reconstruct a unitary matrix as (4), i.e. to see if the data are compatible with the theoretical model. The compatibility conditions are better understood in the frame of a phenomenological model, which in our case will be a relationship between the theoretical object (4) and the experimental data (5). The theoretical tool we have at our disposal for defining such a relationship is the unitarity property of the CKM matrices. The aim of any phenomenological analysis is at least twofold: a) checking the consistency of data (5) with the theoretical model (4), and b) determination of theoretical parameters s_{ij} and δ from the experimental data (5) if the data and the theoretical model are compatible. In other words our scope is the reconstruction of the unitary matrix (4) from experimental data. In the following we present two phenomenological models: unitarity condition method and unitarity triangle method.

In both the models we use the embedding of unitary matrices into the set of double stochastic matrices, see

(3), implying that the following relations should hold

$$\begin{aligned} \sum_{i=d,s,b} |U_{ji}|^2 - 1 &= \sum_{i=d,s,b} V_{ji}^2 - 1 = 0, \quad j = u, c, t \\ \sum_{i=u,c,t} |U_{ij}|^2 - 1 &= \sum_{i=u,c,t} V_{ij}^2 - 1 = 0, \quad j = d, s, b \end{aligned} \quad (6)$$

In the above relations V_{ij} are numbers affected by errors obtained from experiments, and the exact fulfilment of Eqs.(6) is not at all guaranteed in any phenomenological analysis. Even if they are exactly satisfied, the unitarity property could not be satisfied. And this is one main reason for using different notations for experimental moduli V_{ij} , and for theoretical moduli $|U_{ij}|$. The distinction is also necessary since the phenomenologists are not aware of the natural embedding of unitary matrices, through relation (3), into a larger set, that of double stochastic matrices (2), i.e. they are not yet aware of the necessity to find a criterion for the separation of unitary matrices within the double stochastic set, only the intersection of the two sets being relevant for the electroweak physics phenomenology.

A. Unitarity condition method

The specific condition defining this model is the fulfilment of the relation

$$V = |U|^2$$

between the theoretical object (4) and the experimental data (5), relation that has to be understood as working entry wise leading to the following relations

$$\begin{aligned} V_{ud}^2 &= c_{12}^2 c_{13}^2, \quad V_{us}^2 = s_{12}^2 c_{13}^2, \quad V_{ub}^2 = s_{13}^2 \\ V_{cb}^2 &= s_{23}^2 c_{13}^2, \quad V_{tb}^2 = c_{13}^2 c_{23}^2, \\ V_{cd}^2 &= s_{12}^2 c_{23}^2 + s_{13}^2 s_{23}^2 c_{12}^2 + 2s_{12}s_{13}s_{23}c_{12}c_{23}\cos\delta, \\ V_{cs}^2 &= c_{12}^2 c_{23}^2 + s_{12}^2 s_{13}^2 s_{23}^2 - 2s_{12}s_{13}s_{23}c_{12}c_{23}\cos\delta, \quad (7) \\ V_{td}^2 &= s_{13}^2 c_{12}^2 c_{23}^2 + s_{12}^2 s_{23}^2 - 2s_{12}s_{13}s_{23}c_{12}c_{23}\cos\delta, \\ V_{ts}^2 &= s_{12}^2 s_{13}^2 c_{23}^2 + c_{12}^2 s_{23}^2 + 2s_{12}s_{13}s_{23}c_{12}c_{23}\cos\delta \end{aligned}$$

This phenomenological model is similar to that proposed by Wolfenstein, [10], i.e. it is a direct relationship between the measured values V_{ij} and the theoretical parameters entering the unitary matrix (4), s_{ij} and δ . The difference between the two approaches is the following: we make no approximations on the right hand of Eqs. (7), and we will explicitly make use of the double stochasticity relations (6). One sees also that on the left hand in Eqs. (7) there are sets of *nine* numbers, V_{ij}^2 , obtained from experiments, and on the right hand enter only *four* independent parameters. Hence the consistency problem of the equations (7) is a natural one, and it has to be resolved. An other remark is that the δ dependence of equations (7) is through the cosine function, which is an even function, so we can restrict the range of δ to $(0, \pi)$,

when we use as independent parameters the moduli V_{ij} , without loss of generality.

The main problem to be solved is to find in what conditions the system (7) could have a physical solution, for arbitrary numbers V_{ij} satisfying Eqs.(6), i.e. from the full set of double stochastic matrices. In Ref. [1] it was shown that the necessary and sufficient conditions the data have to satisfy in order to the matrix (5) comes from a unitary matrix are

$$0 \leq s_{ij} \leq 1, \quad \text{and} \quad -1 \leq \cos\delta \leq 1 \quad (8)$$

when s_{ij} and $\cos\delta$ are found by resolving the equation system (7). The most constraining condition is the second one, i.e. $\cos\delta \in (-1, 1)$, which is the separation criterion between the double stochastic and unitary matrices, whose fulfilment is compulsory. In fact for the 3×3 matrices, if the relations (6) are satisfied, there always exists a solution for s_{ij} which is physical, and, e.g., it could be obtained by using three independent relations from the first five appearing in equations (7). The relations (8) are the consistency conditions between the data and the theoretical model.

To see the constraining power of the above relations (8) we assume for the moment that the relations (6) are exactly satisfied. The matrix (4) depends on *four* independent parameters and we choose them as four independent moduli, i.e. four $|U_{ij}|$. Now we use the relations (6) and the entries from the data set (5) to construct a matrix, $S = (S_{ij})$, whose square entries form a double stochastic matrix. In order to simplify the formulae form we make the notation

$$\begin{aligned} V_{ud} &= a, \quad V_{us} = b, \quad V_{ub} = c \\ V_{cd} &= d, \quad V_{cs} = e, \quad V_{cb} = f \end{aligned} \quad (9)$$

For example, if the four independent parameters are the moduli $V_{us} = b, V_{ub} = c, V_{cd} = d, V_{cb} = f$, the matrix $S^2 = (S_{ij}^2)$, where

$$S = (S_{ij}) = \begin{pmatrix} \sqrt{1-b^2-c^2} & b & c \\ d & \sqrt{1-d^2-f^2} & f \\ \sqrt{b^2+c^2-d^2} & \sqrt{d^2+f^2-b^2} & \sqrt{1-c^2-f^2} \end{pmatrix} \quad (10)$$

is double stochastic. The above choice is motivated by the fact that the moduli b, c and f are the essential parameters used by the CKM fitter groups. By using the relations $V_{ij}^2 = S_{ij}^2$ in equations (7) one gets values for s_{ij} and $\cos\delta$ as follows

$$\begin{aligned} s_{12}^{(1)} &= \frac{V_{us}}{\sqrt{1-V_{ub}^2}} = \frac{b}{\sqrt{1-c^2}}, \\ s_{23}^{(1)} &= \frac{V_{cb}}{\sqrt{1-V_{ub}^2}} = \frac{f}{\sqrt{1-c^2}}, \quad s_{13}^{(1)} = V_{ub} = c \quad (11) \\ \cos\delta^{(1)} &= \frac{d^2(1-c^2)^2 - b^2(1-c^2) + f^2(b^2-c^2 + c^2(b^2+c^2))}{2bcf\sqrt{1-b^2-c^2}\sqrt{1-c^2-f^2}} \end{aligned}$$

By doing similar computations, but now with the independent parameters a , b , d , e , entering in a double

stochastic matrix similar to (10), one gets

$$\begin{aligned} s_{12}^{(2)} &= \frac{b}{\sqrt{a^2 + b^2}}, \quad s_{13}^{(1)} = \sqrt{1 - a^2 - b^2}, \quad s_{23}^{(2)} = \frac{\sqrt{1 - d^2 - e^2}}{\sqrt{a^2 + b^2}} \\ \cos \delta_2^{(2)} &= \frac{b^2(1 - b^2) - a^2(1 - a^2) + d^2(a^2 - b^2 + b^2(a^2 + b^2)) - e^2(a^2(a^2 - 1) + b^2(1 + a^2))}{2ab\sqrt{1 - a^2 - b^2}\sqrt{1 - d^2 - e^2}\sqrt{a^2 + b^2 + d^2 + e^2 - 1}} \end{aligned} \quad (12)$$

and so on. Now we provide two numerical matrices, built as in (10), whose squared moduli are double stochastic.

$$\begin{aligned} M_1 &= \begin{pmatrix} \frac{993907}{10^6} & \frac{22691}{10^5} & \frac{\sqrt{17007251}}{10^6} \\ \frac{2269}{10^4} & \frac{24327}{25000} & \frac{3\sqrt{453251}}{50000} \\ \frac{\sqrt{21545351}}{10^6} & \frac{\sqrt{16271665}}{10^5} & \frac{\sqrt{998351289149}}{10^6} \end{pmatrix} \\ &\approx \begin{pmatrix} 0.973907 & 0.22691 & 0.004124 \\ 0.2269 & 0.97308 & 0.040394 \\ 0.004642 & 0.040338 & 0.999175 \end{pmatrix} \end{aligned} \quad (13)$$

$$\begin{aligned} M_2 &= \begin{pmatrix} \frac{973907}{10^6} & \frac{22691}{10^5} & \frac{\sqrt{17007251}}{10^6} \\ \frac{2267}{10^4} & \frac{9\sqrt{7306909}}{25000} & \frac{3\sqrt{453251}}{50000} \\ \frac{\sqrt{112265351}}{10^6} & \frac{\sqrt{15364455}}{10^5} & \frac{\sqrt{998260569149}}{10^6} \end{pmatrix} \\ &\approx \begin{pmatrix} 0.973907 & 0.22691 & 0.004124 \\ 0.2267 & 0.973127 & 0.04039 \\ 0.010596 & 0.039197 & 0.999175 \end{pmatrix} \end{aligned} \quad (14)$$

The moduli entering the matrices M_1 and M_2 have been chosen such that they should not be too far from the recommended moduli values given by the PDG 2006 data, [9], i.e. within the error corridors, and not too far from the central values of the last fit by CKM fitter Group [11]. Their difference is

$$M_1 - M_2 \approx \begin{pmatrix} 0 & 0 & 0 \\ 0.0002 & -4.7 \times 10^{-5} & 0 \\ -0.00595 & 0.00114 & 0 \end{pmatrix} \quad (15)$$

i.e. the entry moduli of the matrix (15) are much small than the current experimental errors on the corresponding moduli. In fact only the modulus $M_1(2,1)$ was changed by an amount of 2×10^{-4} and this change was propagated to get again a double stochastic matrix.

By using the formulae (11) and (12) one gets from M_1

$$\begin{aligned} s_{12}^{(i)} &= \frac{226910}{\sqrt{999982992749}} \approx 0.226912, & s_{13}^{(i)} &= \frac{\sqrt{17007251}}{10^6} \approx 0.004124, \\ s_{23}^{(i)} &= 60 \sqrt{\frac{453251}{999982992749}} \approx 0.040395, & i &= 1, 2, \dots \\ \cos \delta^{(i)} &= \frac{613855254083724801174896537}{64224997110656250\sqrt{8008162692156206007509}} \approx 1.068 \end{aligned} \quad (16)$$

The first remark is that s_{ij} and $\cos \delta$ take the same values when the data come from an exact double stochastic matrix, i.e. they are independent of the four independent moduli we choose to parameterise the data. This is a consequence of the fact that the double stochasticity (unitarity) properties do not change if we interchange the columns and/or the rows between themselves, or we use

the transposed, or the complex conjugated matrix. The second remark is that, although the entries of the matrix M_1 are good from an experimental point of view, they are not good from a theoretical point of view since $\cos \delta$ is unphysical.

From the second matrix, M_2 , we get

$$s_{12}^{(i)} = \frac{226910}{\sqrt{999982992749}}, \quad s_{13}^{(i)} = \frac{\sqrt{17007251}}{10^6}, \quad s_{23}^{(i)} = 60\sqrt{\frac{453251}{999982992749}}, \quad i = 1, 2, \dots \quad (17)$$

$$\cos \delta^{(i)} = -\frac{1897412766859734762379196101}{128449994221312500\sqrt{8008162692156206007509}} \approx -0.165$$

From a theoretical point of view the second matrix, M_2 , is compatible with unitarity. Please remark that in both the cases the mixing angles s_{ij} take identical values although only M_2 comes from a unitary matrix. Hence the separation criterion between unitary matrices and purely double stochastic ones is provided by the physical values taken by $\cos \delta$. By looking at the difference matrix $M_1 - M_2$, Eq.(15), one sees that the moduli differ by numbers which are smaller than the experimental errors and, in spite of this, $\cos \delta$ has a substantial jump from a unphysical value, 1.068, to a physical one, -0.165, which is equivalent to $\delta = 99.5^\circ$. Thus the compatibility condition between data and theoretical model, $-1 \leq \cos \delta \leq 1$, puts very strong conditions on data in order to they come from a unitary matrix, and, as the above example shows, the condition is very sensitive even to small moduli variation.

When the relations (6) are exactly satisfied the reconstruction algorithm of a unitary matrix from a double stochastic one is the following: start with a double stochastic matrix as M_1^2 , or M_2^2 , solve the equations (7) and obtain results as those given by equations (16) and (17). If the numerical value for $\cos \delta$ satisfies the inequalities (8), then with the values for s_{ij} and $\cos \delta$ go to equation (4) and find the corresponding unitary matrix. In the above cases only the numerical results from M_2 are compatible to the existence of a unitary matrix, while those from the M_1 matrix are not, although the matrix M_1 is a double stochastic one, and from the usual phenomenological point of view both data could be considered as being “physical”. The reconstruction of the unitary matrix from the M_2 data is the matrix \mathcal{U} which is obtained by the substitution of numerical values for s_{ij} and $\cos \delta$ given by Eqs. (17) into formula (4).

The true real case is when the double stochasticity relations (6) are not exactly satisfied, and by consequence the double stochastic matrices, built such as the matrix S , are different. To see what happens in this case we will use only the central values from the PDG 2006 data [9], which are

$$\begin{aligned} a &= 0.97377, & b &= 0.2257, & c &= 4.31 \times 10^{-3}, \\ d &= 0.230, & e &= 0.957, & f &= 41.6 \times 10^{-3} \end{aligned} \quad (18)$$

and those from the fit [12], namely

$$\begin{aligned} a &= 0.97504, & b &= 0.2221, & c &= 3.505 \times 10^{-3}, \\ d &= 0.222, & e &= 0.97422, & f &= 40.8 \times 10^{-3} \end{aligned} \quad (19)$$

With them we form the double stochastic matrices corresponding to the phenomenological models (11) and, respectively, (12), and one gets in the first case

$$\cos \delta^{(1)} = 25.98, \quad \cos \delta^{(2)} = 1.58 \quad (20)$$

and

$$\cos \delta^{(1)} = 0.6, \quad \cos \delta^{(2)} = -0.377 i \quad (21)$$

in the second case. If in the first case the results could be considered as being “normal” since the moduli were obtained by doing some weighted means on moduli data obtained by experimenters, in the second case the numbers were obtained from a fit which used the unitarity triangle approach, formalism that is supposed to take properly into account the unitarity property. Unfortunately this does not happens, the second form for $\cos \delta$ providing an imaginary value. As we will show in the next section this is a characteristic of this fit, not an accident. The corresponding values for $\cos \delta$ are different because, e.g., in the first case, the parameters, a, b, c, d, e and f do not come from the same doubly stochastic matrix, e.g., $f \neq \sqrt{1 - d^2 - e^2}$, or numerically, $0.0416 \neq 0.1768$, and so on. Thus even in the case when the relations (6) are exactly satisfied, by choosing four independent moduli and constructing with them a double stochastic matrix, the numerical results show that unitarity could not be satisfied, many $\cos \delta$ values being not physical, and these values depend on the *chosen four independent moduli*, although the solution for $\cos \delta$ of Eqs. (7) must be unique. Hence we have to find a solution to this problem.

B. Unitarity triangle method

The second phenomenological model is defined by the orthogonality relations of rows, and, respectively columns of a unitary matrix, *together* with the double stochasticity relations (6). The last condition was never used in the previous approaches, see Ref.[13], or [14]. Although there are six such relations usually one considers only the orthogonality of the first and the third columns of U , relation that is written as

$$U_{ud}U_{ub}^* + U_{cd}U_{cb}^* + U_{td}U_{tb}^* = 0 \quad (22)$$

where $*$ denotes the complex conjugation. The above relation can be visualised as a triangle in the complex plane, and usually it is scaled by dividing through the

middle term such that the length of one side is 1. In fact one may divide by any other term because all the three possible triangles are *similar*, i.e. they have the *same angles*. On the other hand all the six triangles obtained from all orthogonality relations such as (22) are *equivalent*, i.e. they have the same *area*, A , and furthermore the relation $J = 2A$ holds, where J is the Jarlskog invariant, see [2]-[3]. The side lengths of these triangles are auxiliary parameters, and have no physical significance. In contradistinction to sides, the angles are measurable quantities such that they are important from an experimental point of view, see in this respect Ref. [14]. By taking into account the relation (10), the other sides of the triangle (22) have the lengths

$$R_b^{(1)} = \left| \frac{U_{ud}U_{ub}^*}{U_{cd}U_{cb}^*} \right| = \frac{c\sqrt{1-b^2-c^2}}{df}, \quad (23)$$

$$R_t^{(1)} = \left| \frac{U_{td}U_{tb}^*}{U_{cd}U_{cb}^*} \right| = \frac{\sqrt{b^2+c^2-d^2}\sqrt{1-c^2-f^2}}{df}$$

and, respectively,

$$R_b^{(2)} = \left| \frac{U_{ud}U_{ub}^*}{U_{cd}U_{cb}^*} \right| = \frac{a\sqrt{1-a^2-b^2}}{d\sqrt{1-d^2-e^2}}, \quad (24)$$

$$R_t^{(2)} = \left| \frac{U_{td}U_{tb}^*}{U_{cd}U_{cb}^*} \right| = \frac{\sqrt{1-a^2-d^2}\sqrt{a^2+b^2+d^2+e^2-1}}{d\sqrt{1-d^2-e^2}}$$

Since the exact fulfilment of Eqs.(6) does not mean the fulfilment of the unitarity property, as our numerical results (16-17) show, that property is implemented in this approach by the conditions: all ratios should be positive, $R_a \geq 0$, $a = b, t, \dots$, and the following inequalities should be satisfied [3]

$$|R_b^{(i)} - R_t^{(i)}| \leq 1 \leq R_b^{(i)} + R_t^{(i)}, \quad i = 1, 2, \dots \quad (25)$$

that are equivalent with the conditions (8). Indeed by computing the above ratios by using matrices M_1 and M_2 one gets

$$R_b^{(1)} = 0.438, \quad R_t^{(1)} = 0.506, \quad (26)$$

$$R_b^{(2)} = 1.156, \quad R_t^{(2)} = 0.438$$

such that the relations (25) give

$$0.068 \leq 1 \leq 0.944, \quad 0.718 \leq 1 \leq 1.594 \quad (27)$$

showing the perfect (theoretical) equivalence at this level of both the phenomenological models. If now we compute the above ratios by using moduli from the fit [12] one finds

$$R_b^{(1)} = 0.377, \quad R_t^{(1)} = 0.83, \quad (28)$$

$$R_b^{(2)} = 0.613i, \quad R_t^{(2)} = 0.404$$

in perfect agreement with the results (21), showing the inconsistency of the fit.

The Eqs.(23-24) represent the correct form for the side lengths of the “standard” unitarity triangle. Its angles are easily related to the phases of U_{cd} and U_{td} .

The other unitarity triangle which will provide another two independent angles is given by the orthogonality of the second and the third columns

$$U_{us}U_{ub}^* + U_{cs}U_{cb}^* + U_{ts}U_{tb}^* = 0 \quad (29)$$

One gets similarly

$$R_c^{(1)} = \left| \frac{U_{us}U_{ub}^*}{U_{cs}U_{cb}^*} \right| = \frac{bc}{f\sqrt{1-d^2-f^2}}, \quad (30)$$

$$R_s^{(1)} = \left| \frac{U_{ts}U_{tb}^*}{U_{cs}U_{cb}^*} \right| = \frac{\sqrt{1-c^2-f^2}\sqrt{d^2+f^2-b^2}}{f\sqrt{1-d^2-f^2}}$$

and, respectively

$$R_c^{(2)} = \left| \frac{U_{us}U_{ub}^*}{U_{cs}U_{cb}^*} \right| = \frac{b\sqrt{1-a^2-b^2}}{e\sqrt{1-d^2-e^2}}, \quad (31)$$

$$R_s^{(2)} = \left| \frac{U_{ts}U_{tb}^*}{U_{cs}U_{cb}^*} \right| = \frac{\sqrt{1-b^2-e^2}\sqrt{a^2+b^2+d^2+e^2-1}}{e\sqrt{1-d^2-e^2}}$$

Computing the above ratios with the same numbers as in (28) one gets

$$R_c^{(1)} = 0.02, \quad R_s^{(1)} = 1.01, \quad (32)$$

$$R_c^{(2)} = 0.038i, \quad R_s^{(2)} = 1.01$$

showing the same phenomenon as that given by the relations (28) concerning the implementation of unitarity constraints for this triangle.

Similar to the first case, two angles of the triangle generated by the relation (29) are directly connected to the phases of the elements U_{cs} and U_{ts} . From the above relations we may obtain all the angles of the two triangles generated by the relations (22), and, respectively, (29).

For getting the relationship between the phases of complex entries of the CKM matrix (4) and the moduli and phase δ we make the following notation $U_{kl} = \pm|U_{kl}|e^{i\omega_{kl}}$, $k = 2, 3$, $l = 1, 2$, where the minus sign will be taken in front of $|U_{21}|$ and $|U_{32}|$. We make full use of the property that in our parameterisation (4) of the CKM matrix, the entries U_{ud} , U_{us} , U_{ub} , U_{cb} and U_{tb} are all real quantities, and of the invariance of unitarity triangles angles with respect to the scaling factor. Thus we write the relations (23) in a complex form as

$$\frac{\cos\omega_{21} + i\sin\omega_{21}}{R_b} = -\frac{U_{cd}U_{cb}^*}{U_{ud}U_{ub}^*}, \quad (33)$$

$$\frac{(\cos\omega_{31} + i\sin\omega_{31})R_t}{R_b} = \frac{U_{td}U_{tb}^*}{U_{ud}U_{ub}^*}$$

Please remark that the phases ω_{21} and ω_{31} coincide with the phase of $-U_{cd}$, and, respectively, U_{td} . The above phases coincides, in some cases modulo π , with the angles of the unitarity triangle obtained by scaling the relation

(22) dividing by the first term, i.e. $U_{ud}U_{ub}^*$. Since the triangles are similar the following relations hold $\omega_{21} = \gamma$, $\omega_{31} = \alpha$. From the first relation (33) we easily obtain

$$\begin{aligned}\sin \omega_{21} &= \frac{s_{12} s_{23} c_{23} \sin \delta R_b}{s_{13} c_{12}}, \\ \cos \omega_{21} &= \frac{s_{23}(s_{13} s_{23} c_{12} + s_{12} c_{23} \cos \delta) R_b}{s_{13} c_{12}}\end{aligned}\quad (34)$$

where from one gets

$$\tan \omega_{21} = \frac{s_{12} c_{23} \sin \delta}{s_{13} s_{23} c_{12} + s_{12} c_{23} \cos \delta} \quad (35)$$

The above formula for $\tan \omega_{21}$ depends only on theoretical parameters entering (4), and does not depend on the lengths of the unitarity triangle. It shows that it is not necessary the use of unitarity triangles construction to obtain their angles, they can be easily obtained from the moduli V_{ij} , through relations such as (11) and (12).

From relations (34) we get a direct formula for $\cos \delta$. Indeed, from the identity $\sin^2 \omega_{21} + \cos^2 \omega_{21} = 1$ we find

$$\cos \delta = \frac{s_{13}^2 c_{12}^2 - s_{23}^2 (s_{12}^2 c_{23}^2 + s_{13}^2 s_{23}^2 c_{12}^2) R_b^2}{2 s_{12} s_{13} s_{23}^3 c_{12} c_{23}} \quad (36)$$

that depends on s_{ij} and R_b . If in it we substitute the formulae for s_{ij} from relations (11), and R_b from relations (23), we find the expression given in Eq.(11) for $\cos \delta$, that shows at this level the perfect equivalence between the two approaches. Hence there are explicit formulae for $\cos \delta$ in terms of triangle lengths and mixing angles s_{ij} .

Doing similar calculations by starting with the second Eq.(33) one gets

$$\tan \omega_{31} = \frac{s_{12} s_{23} \sin \delta}{-s_{13} c_{12} c_{23} + s_{12} s_{23} \cos \delta} \quad (37)$$

and, respectively,

$$\cos \delta = \frac{-s_{13}^2 c_{12}^2 R_t^2 / R_b^2 + c_{23}^2 (s_{12}^2 s_{23}^2 + s_{13}^2 c_{23}^2 c_{12}^2)}{2 s_{12} s_{13} s_{23} c_{12} c_{23}^3} \quad (38)$$

In conclusion the orthogonality relation (22) provides only two independent angles, that depend on *four* independent parameters, s_{12} , s_{13} , s_{23} , and δ , and for a complete determination of parameters entering a unitary matrix we have to use another unitarity triangle, for example that generated by the relation (29). By doing similar computations one gets

$$\tan \omega_{22} = \frac{c_{12} c_{23} \sin \delta}{-s_{12} s_{13} s_{23} + c_{12} c_{23} \cos \delta}, \quad (39)$$

$$\tan \omega_{32} = \frac{c_{12} s_{23} \sin \delta}{s_{12} s_{13} c_{23} + c_{12} s_{23} \cos \delta} \quad (40)$$

Hence we can write the matrix (4) under the form

$$U = \begin{pmatrix} U_{ud} & U_{us} & U_{ub} \\ -|U_{cd}|e^{i\omega_{21}} & |U_{cs}|e^{i\omega_{22}} & U_{cb} \\ |U_{td}|e^{i\omega_{31}} & -|U_{ts}|e^{i\omega_{32}} & U_{tb} \end{pmatrix} \quad (41)$$

and we call ω_{ij} the fundamental phases since with them one gets the angles of *all* unitarity triangles. The phases entering the matrix (41) corresponding to the moduli matrix M_2 are given by

$$\omega_{21} = 99.46^\circ, \quad \omega_{31} = 121.427^\circ, \quad (42)$$

$$\omega_{22} = 99.50^\circ, \quad \omega_{32} = 98.153^\circ \quad (43)$$

Our proposal is to use the matrix form (41) in all phenomenological analyses, and the experimenters have to measure all its phases which really appear as parameters in many electroweak processes. For a similar proposal see Ref. [4].

The important conclusions of this subsection are: a) for a complete determination of a unitary matrix from data when one uses the unitarity triangle model one needs the use of at least two unitarity triangles if one wants to obtain reliable results, b) the use of double stochasticity relations (6) is compulsory, otherwise the fit gets senseless, and c) checking of fit results to see if they are invariant with respect to the choice of four independent moduli used to parametrise the data. The numerical results, provided by the relations (21), (28) and (32), show that all the above requirements were not fulfilled by the fit [12].

C. Recovery of unitary matrices from error affected data

Until now we supposed that the double stochasticity relations (6) were exactly satisfied, i.e. the data were not affected by errors. Even in this case by using *measured*, or *fit determined* moduli, and by forming with them double stochastic matrices we got that the numerical computations on $\cos \delta$ and lengths of unitarity triangles lead to unphysical results, see Eqs. (20-21), (26-28) and (32). By consequence we have to see how the reconstruction algorithm which works for data coming from exact double stochastic matrices has to be modified in order to provide reliable results in the presence of errors. The experimental data on moduli do not satisfy the double stochasticity relations (6), hence the first condition which must be imposed when doing a fit is that these relations should be satisfied with a great precision, otherwise one cannot speak of unitarity fulfilment, because the recovery process gets senseless. Since the double stochastic matrices obtained from data by using different groups of four independent moduli are really different, leading to different values for $\cos \delta$, see e.g. Eqs.(20-21), we have to impose that all $\cos \delta$ values should be (approximately) the same. On the other hand the explicit form for $\cos \delta$ depends on the four independent moduli we choose to parameterise the data, hence in analysing the data we have to make full use of the axiom 2a) from Introduction.

The recovery method of unitary matrices from data that we expose in the following is a least squares method for checking the compatibility of data with the theoretic

cal models in both the approaches, and stresses the necessity that the χ^2 -function have to contain two kind of terms: the first has to impose the fulfilment of unitarity constraints, and the second should take into account the physical quantities measured in experiments. The piece of unitarity constraints has an independent part provided by the double stochasticity relations, and a dependent one upon the phenomenological model. Constraints implied by double stochasticity relations are written as

$$\chi_{ds}^2 = \sum_{j=u,c,t} \left(\sum_{i=d,s,b} V_{ji}^2 - 1 \right)^2 + \sum_{j=d,s,b} \left(\sum_{i=u,c,t} V_{ij}^2 - 1 \right)^2 \quad (44)$$

The constraints generated by the unitarity condition method are given by

$$\chi_1^2 = \sum_{i < j} (\cos \delta^{(i)} - \cos \delta^{(j)})^2, \quad -1 \leq \cos \delta^{(i)} \leq 1 \quad (45)$$

A similar formula one gets for the second phenomenological model where instead of $\cos \delta$ one uses the side lengths R_a , see Ref.[1]. The third component which takes into account the experimental data has the form

$$\chi_2^2 = \sum_i \left(\frac{d_i - \tilde{d}_i}{\sigma_i} \right)^2 \quad (46)$$

where d_i are the theoretical functions one wants to be found from fit, \tilde{d}_i is the numerical matrix that describes the corresponding experimental data, while σ is the matrix of errors associated to \tilde{d}_i . Hence a χ^2 -test could be the function

$$\chi^2 = \chi_{ds}^2 + \chi_1^2 + \chi_2^2 \quad (47)$$

which will be used in numerical computations.

The compatibility condition of Eqs. (7), together with the invariance of physical quantities with respect of the four independent moduli group chosen to parametrise the data, imply strong conditions on all the moduli given by the relations $\cos \delta^{(i)} \approx \cos \delta^{(j)}$ for $i \neq j$, see Eq. (45), which are not easy to implement in a fit, but gives us at least one reward: a method for doing statistics on unitary matrices.

It is well known that the problem of doing statistics on (moduli) of unitary matrices was an open problem, see e.g. [15]. The embedding relations (3) suggest that the right quantities to look at them are the square moduli, $|U_{ij}|^2$. Indeed the convexity property of double stochastic matrices implies that if we have a set of unitary matrices U_1, \dots, U_n then

$$M^2 = \sum_{i=1}^{i=n} x_i \cdot |U_k|^2, \quad \sum_{i=1}^{i=n} x_i = 1, \quad (48)$$

$$0 \leq x_i \leq 1, \quad i = i, \dots, n$$

is a double stochastic matrix. In other words we have to do statistics on the set of double stochastic matrices generated by unitary ones. Thus this property allows us to calculate correctly mean values, $\langle M \rangle$, and error matrices, σ_M , for a set of double stochastic ones, and by consequence for unitary matrices, as follows

$$\langle M \rangle = \sqrt{\left(\sum_{k=1}^{k=n} |U_k|^2 \right) / n}, \quad (49)$$

$$\sigma_M = \sqrt{\left(\sum_{k=1}^{k=n} |U_k|^4 \right) / n - \langle M \rangle^4}$$

where the square roots are taken entry wise. If the mean value matrix obtained in this way is not too far from a unitary one, one can reconstruct from it an (approximate) unitary matrix. Also important is the relationship between the “central” values matrix and those obtained by adding the error matrix σ_M . Thus the above formulae suggest that the $\pm 3\sigma$ matrices should be calculated by the formulae

$$M_+ = \sqrt{M^2 + 3\sigma_M^2}, \quad M_- = \sqrt{M^2 - 3\sigma_M^2} \quad (50)$$

In this way we provided a complete formalism for doing fits on data from the electroweak sector, which is a very robust one such as can be seen in the next Section.

III. TESTING DATA AND FITS

In the following we want to show what are the main consequences of the above formalism, especially what are the subtleties and the results obtained when doing statistics on unitary matrices by the above method. For that we will make use of all 165 different forms for $\cos \delta$. We test the Reviews of Particle Physics 2004 [16] and 2006 data [9], the lattice computations of CKM moduli [17], and the published fits [11] and [12].

1. 2004 PDG data [16]. As it is well known there one finds upper and lower bounds for each modulus, at 95% confidence level. We used the mean values, calculated as half the sum of upper and lower bounds, and the corresponding σ_i computed as half the difference of the bounds. One gets that the double stochasticity is quite well satisfied by the mean values, all the six sums differing from 1 by amounts of order $10^{-6} - 10^{-4}$, which lead to

$$\langle \cos \delta \rangle = 0.063 + 0.008i \quad (51)$$

Because the mean real and imaginary parts are small, the above results suggest that the central data are consistent with a δ value around 90° , and a fit done by using our method provided the numerical matrix

$$V_1 = \begin{pmatrix} 0.9748379 & 0.222885 & 0.00365667 \\ 0.222699 & 0.974025 & 0.0409987 \\ 0.0098077 & 0.0399759 & 0.999153 \end{pmatrix} \quad (52)$$

which leads to the following value

$$\delta = (90.01 \pm 0.41)^\circ \quad (53)$$

For the above matrix the double stochasticity property is well satisfied, all six relations (6) taking values whose magnitude is of the order of 10^{-6} , and this implies a small (statistical) error for δ .

If one uses the recommended values [16] for the entries on the first two rows, and the mean values for moduli on the third row one gets big non-physical values

$$\langle \cos \delta \rangle = -24.15 - 8.16i, \quad \sigma_{\cos \delta} = 504.21 - 0.39i \quad (54)$$

which show that the recommended values are quite far from values compatible with unitarity requirements.

2. 2006 PDG data [9]. We made a data modification for one single modulus, V_{tb} , by taking its central value equal to $V_{tb} = 0.99912$, instead of $V_{tb} = 0.77^{+0.18}_{-0.24}$, since we saw that the obtained results are similar, and one gets

$$\langle \cos \delta \rangle = 21.557 - 21.320i, \quad \sigma_{\cos \delta} = 62.738 + 7.326i \quad (55)$$

which shows that the mean data are far from being compatible with unitarity. A fit with our formula (47) around the central values from PDG 2006 data provides the matrix

$$V_2 = \begin{pmatrix} 0.974290 & 0.225260 & 0.004114 \\ 0.225064 & 0.973579 & 0.038602 \\ 0.010256 & 0.037441 & 0.999246 \end{pmatrix} \quad (56)$$

where from we get

$$\delta = (101.18 \pm 0.18)^\circ \quad (57)$$

The 2006 PDG data [9] bring an innovation: entering of global fits results into Review of Particle Physics book, see Ceccucci et al. contribution to CKM quark-mixing matrix [18] in the book. The fits taken into account are [11] and [19]. By using the central values provided there one finds

$$\langle \cos \delta \rangle = 0.5084, \quad \sigma_{\cos \delta} = 0.1002 \quad (58)$$

which is equivalent to

$$\delta = (59.44^{+6.46}_{-6.93})^\circ \quad (59)$$

which shows at the level of 1σ an important statistical spreading caused by a poor fulfilment of the double stochasticity relations for the first row and the first column. If we look at the central values $V_c + \sigma$, and respectively $V_c - \sigma$, where $+\sigma$ and $-\sigma$ are slightly different, see Eq.(11.26) in [18], we get

$$\langle \cos \delta^+ \rangle = 1.4175 + 0.0508i, \quad (60)$$

$$\sigma_{\cos \delta^+} = 6.1236 - 0.0118i \quad (61)$$

and respectively

$$\langle \cos \delta^- \rangle = 0.3076 + 0.2324i, \quad (62)$$

$$\sigma_{\cos \delta^-} = 6.2063 - 0.0115i \quad (63)$$

Both results show that moduli values $V_c \pm \sigma$ have no physical relevance and provide one numerical example showing the shortcomings of the unitarity triangle approach such as it is used in the present day. The big values for $\sigma_{\cos \delta^\pm}$ confirm that many $\cos \delta$ values take values outside the physical range $(-1, 1)$, and an important number takes imaginary values. For example in the second case, $V_c - \sigma$, the real values are within the interval $\cos \delta \in (-16.45, 19.17)$, and the imaginary ones, 19 values from 165, within the interval $\cos \delta \in (-0.355, 7.525)i$

In our opinion this example shows clearly the two causes which generate such results: a) fitting only with a single unitarity triangle and using essentially only three moduli, b , c and f , and b) non existence of a sound procedure for doing statistics on moduli of unitary matrices.

3. By using the lattice results from Ref. [17] on CKM matrix moduli one gets

$$\langle \cos \delta \rangle = 8.962 - 11.315i, \quad \sigma_{\cos \delta} = 20.15 + 5.03i \quad (64)$$

which shows that lattice computations are still far from results compatible with unitarity constraints. The above results show that the double stochasticity property is not satisfied, hence the numerical effort has to be done to improve it. By looking for a compatible unitary matrix around the mean moduli values from Ref. [17], with our method one gets the numerical matrix

$$V_3 = \begin{pmatrix} 0.974330 & 0.225086 & 0.0041314 \\ 0.225003 & 0.973562 & 0.0393922 \\ 0.007401 & 0.038911 & 0.999215 \end{pmatrix} \quad (65)$$

which leads to $\delta = (56.04 \pm 0.62)^\circ$. This moduli matrix can be used as a good “unitary witness” to improve the numerical algorithms used to obtain results as those given in Ref. [17].

The sensitivity of our method to small moduli variations can be seen by looking at the entries of matrices (56) and (65). The difference of the moduli on the first two rows is of the order of 10^{-4} , while δ changes from 101° to 56° .

4. CKM fitter Group 2001 results [12]. The real and imaginary values for $\cos \delta$ are within the bounds $\cos \delta \in (-1.082, 1.596)$, respectively, $\cos \delta \in i(-2.431, 0.835)$ and one gets

$$\langle \cos \delta \rangle = 0.373 - 0.087i, \quad \sigma_{\cos \delta} = 0.358 + 0.907i \quad (66)$$

Hence one could say that the central values together with their $\pm\sigma$ companions are not compatible with unitarity. This happened because the invariance of the physical results with respect to all the choices for the *four* independent moduli groups, and the double stochasticity relations were not yet implemented in the unitarity triangle model, and, by consequence, the fit have no physical significance.

5. CKM fitter Group 2005 results [11]. The last published results by CKM fitter Group are, in our opinion, the best published results until now. Thus our fourth

selected matrix is given by the central values from [11], see their Table 3

$$V_4 = \begin{pmatrix} 0.97400 & 0.2265 & 0.00387 \\ 0.2264 & 0.97317 & 0.04113 \\ 0.00826 & 0.04047 & 0.999146 \end{pmatrix} \quad (67)$$

The errors on moduli are of the order 5.5×10^{-4} for V_{ud} and V_{cs} , and of order 2.5×10^{-3} for V_{us} and V_{cd} . The double stochasticity is well satisfied obtaining numbers of the order $(10^{-8} - 10^{-6})$. On this matrix one can check how the fulfilment degree of double stochasticity property reflects in $\cos \delta$ values. One gets for δ and Jarlskog invariant, J , the following values

$$\delta = (62.07^{+3.79}_{-3.92})^\circ, \quad J = (3.84^{+0.62}_{-0.79}) \times 10^{-5} \quad (68)$$

We remind that for a double stochastic matrix the statistical error of δ is zero, such that the errors for δ give a measure of the double stochasticity fulfilment. As a curiosity we remark that, from the point of view of unitarity constraints, the numerical results (67) are better than those appearing in PDG 2006 data, [18], perhaps since the last ones were obtained by merging results from two different fitting groups. Unfortunately the errors accompanying the matrix (67) have no physical relevance if are used according with the present day rules.

By doing a fit with our method around the values from Eq.(67) one gets such a matrix, which is only slightly different from the above one. If we check now the results for $V_4 + 1\sigma$, and $V_4 - 1\sigma$, where the matrices $+ \sigma$ and $- \sigma$ are different, see their Table 3, and one adds the errors such as they are given, one gets

$$\begin{aligned} \langle \cos \delta \rangle_+ &= -2.214 - 4.559i, \\ \langle \cos \delta \rangle_- &= -2.91 - 5.33i \end{aligned} \quad (69)$$

If we proceed by using our method, i.e. we calculate δ_\pm by using the moduli obtained with the formulae $\sqrt{V_4^2 \pm \sigma^2}$ one gets

$$\delta_+ = (61.02^{+7.16}_{-7.7})^\circ \quad \text{and} \quad \delta_- = (61.91^{+3.45}_{-3.56})^\circ \quad (70)$$

Also one finds that the $+ \sigma$ error matrix is overestimated since, for example, computations with the moduli matrix $\sqrt{V_4^2 + 3\sigma^2}$ lead to complex values for $\langle \cos \delta \rangle$, namely $\langle \cos \delta \rangle = 0.554 - 0.015i$.

6. Other selected matrices. Around the central values from PDG 2006 data we imposed a supplementary constraint, namely, $V_{td}/V_{ts} = 0.208$, and we got

$$V_5 = \begin{pmatrix} 0.973556 & 0.228407 & 0.00431 \\ 0.228289 & 0.972704 & 0.0416 \\ 0.00851678 & 0.0409463 & 0.999125 \end{pmatrix} \quad (71)$$

and from it one finds $\delta = (63.706 \pm 0.228)^\circ$.

More exotic matrices are: a) around central PDG 2004 data values

$$V_6 = \begin{pmatrix} 0.975282 & 0.220925 & 0.00340635 \\ 0.220702 & 0.974397 & 0.04299 \\ 0.0106931 & 0.0418558 & 0.999066 \end{pmatrix} \quad (72)$$

with

$$\delta = (98.1 \pm 6.4)^\circ$$

b) around PDG 2006 data central values

$$V_7 = \begin{pmatrix} 0.97328 & 0.2295 & 0.0062 \\ 0.229112 & 0.97256 & 0.04044 \\ 0.0153 & 0.037939 & 0.99916 \end{pmatrix} \quad (73)$$

which leads to

$$\delta = (155.7 \pm 13.5)^\circ$$

or c)

$$V_8 = \begin{pmatrix} 0.973514 & 0.228585 & 0.004311 \\ 0.228564 & 0.97264 & 0.0416 \\ 0.005316 & 0.041484 & 0.999125 \end{pmatrix} \quad (74)$$

equivalent to

$$\delta = (6.4 \pm 4.0)^\circ$$

The double stochasticity property is not very well satisfied by the last three exotic matrices. We provided them to see how our method for doing statistics on unitary matrices works. However, from a phenomenological point of view, all the above eight matrices are good, and each one leads to a perfectly acceptable unitary matrix. The above examples show that the obtained unitary matrices depend strongly on the numerical values around one looks for a compatible unitary matrix. Changing them a little bit one gets different values for δ . Hence the data are not known with a sufficient precision in order to tighten significantly the CP violating phase δ .

The novelty brought by the embedding of unitary matrices into the convex set of double stochastic matrices is the following: we can use the double stochasticity property, and from the (approximate) double stochastic matrices V_i , $i = 1 \dots, 9$, where V_9 denotes the approximate numerical form of the M_2 matrix, we get a continuum of (approximate) double stochastic matrices

$$W^2 = \sum_{i=1}^{i=8} x_i V_i^2 + (1 - \sum_{i=1}^{i=8} x_i) V_9, \quad (75)$$

$$0 \leq x_i \leq 1, \quad i = 1, \dots, 8, \quad \sum_{i=1}^{i=8} x_i \leq 1$$

continuum which depends on eight arbitrary parameters x_i . Its consequence is that with the W matrix one can obtain practically any value for δ within the interval $(0^\circ, 180^\circ)$, all of them being relatively on the same footing in what concerns their agreement with the experimental data on the moduli V_{ij} , and with the fulfilment of unitarity constraints.

If we use the formulae (49) to do statistics on the above nine matrices we find at the symmetric point $x_i = 1/9$, $i = 1, \dots, 8$, the matrix

$$W = \begin{pmatrix} 0.974134 & 0.225921 & 0.0042982 \\ 0.225766 & 0.973323 & 0.0409179 \\ 0.009583 & 0.040019 & 0.999153 \end{pmatrix} \quad (76)$$

and its associated error matrix

$$\sigma = \begin{pmatrix} 0.00119 & 0.00118 & 7.4 \times 10^{-6} \\ 0.00118 & 0.00117 & 1.0 \times 10^{-4} \\ 5.7 \times 10^{-5} & 11.68 \times 10^{-5} & 9.88 \times 10^{-5} \end{pmatrix} \quad (77)$$

which provide the following value for δ

$$\delta = (81.12 \pm 1.23)^\circ \quad (78)$$

Although the moduli matrices, V_i , lead to δ values practically on all the physical interval, $(0, 180)^\circ$, the mean value W satisfies the double stochasticity property quite well shown by the relatively small statistical error in formula (77), which leads to a small error for δ , see Eq.(78). Of course the true matrix σ has to be calculated by taking into account all σ_i matrices, by using the same property: one takes the mean of squared σ_i^2 , which lead to a new matrix whose entries are a little bit bigger than those appearing in (77).

The above results show that the present day fits are not able to determine the CKM matrix entries with a high precision, and by consequence neither δ , nor the J invariant can be well defined. The computations, as well as all the numbers obtained by the so called global fits, use only partial information coming from experiments, which has as consequence the existence of a continuum of unitary matrices compatible with a set of experimental data and unitarity constraints. As the numerical computations show this type of global fit gives better results and in the same time it provides a sound method for doing statistics on unitary matrices.

IV. CONCLUSION

In the paper we have shown that the compatibility between the moduli data and the theoretical model (4) can

be obtained if and only if three conditions are simultaneously satisfied: a) the double stochasticity relations (6) hold to a level of at least 10^{-6} , b) the physical condition $-1 \leq \cos \delta \leq 1$ is satisfied, and c) the numerical values for $\cos \delta$ do not depend on the four independent moduli chosen to parametrise the data.

Our numerical checking of the two approaches in Section 2 has shown that both the theoretical models, unitarity triangle method and the unitarity condition method, send the same signal concerning the compatibility of data with unitarity requirements. Thus now we have two phenomenological approaches, and both of them have to give results consistent each other.

The numerical implementation of the χ^2 -test, such as (47), is a little bit more complicated in both the approaches when one makes use of the exact form of unitarity constraints. This happens since by looking at the denominators of the relations (11-12), (23-17), (30-31), and those similar to them, one sees that there appear square roots which by modifying a little bit the moduli values, change easily their values from real to imaginary ones. On the other hand by using approximations, as those used in the standard form of unitarity triangle model, one easily finds non physical results. However, as our numerical computations from Section 3 show, the implementation of all the exact unitarity constraints can be done.

Our computations have shown that the phenomenologists are not yet aware of the constraining power of unitarity, this one requiring a moduli matching to an order of $10^{-4} - 10^{-6}$, much lower than the experimental errors, tuning that can not be obtained by using approximate formulae. The influence of errors on the final results is that in case they are not small enough, the double stochasticity property provides us continuum sets of approximate unitary matrices. Combining these results with a correct method for doing statistics on unitary matrices we have obtained a powerful tool for checking unitarity properties of data that will become available at the LHC machine in the next years.

-
- [1] P. Diță, J.Math.Phys. **47** (2006) 083510
 - [2] C. Jarlskog and R. Stora, Phys.Lett. **B 208** (1988) 268
 - [3] G.C.Branco and L.Lavoura, Phys.Lett. **B 208** (1988) 123
 - [4] R. Aleksan, B. Kayser and D. London, Phys.Rev.Lett. **73** (1994) 18
 - [5] E. Blucher, E. de Luca, G. Isidori, and V. Lubicz (conveners), *Proc. of the CKM 2005 - WG1 Workshop*, San Diego, March 2005; hep-ph/0512039
 - [6] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, (Academic Press, New York, 1979), Chapter 2
 - [7] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973)
 - [8] L. L. Chau and W. Y. Keung, Phys.Rev.Lett. **53** (1984) 1802
 - [9] W-M Yao *et al.* [Particle Data Group], J.Phys. **G 33** (2006) 1
 - [10] L. Wolfenstein, Phys.Rev.Lett. **51** 1945 (1983)
 - [11] J. Charles *et al.* [The CKMfitter Group], Eur.Phys.J. **C 41** (2005) 1-131; hep-ph/0406184

- [12] A. Höcker, H. Lacker, S. Laplace, and F. R. Le Diberder, Eur.Phys. J. **C 21** (2001) 225
- [13] A. J. Buras, M. E. Lautenbacher and G. Ostermaier, Phys.Rev. **D 50** (1994) 3433
- [14] *The BaBar Physics Book*, P.H. Harison and H.R. Quinn (eds), SLAC-R-504 (1998)
- [15] S. Placzczynski and M.- H. Schune, hep-ph/9911280
- [16] S. Eidelman *et al.* [Particle Data Group], Phys.Lett. **B 592** (2004) 1
- [17] M. Okamoto, Nucl.Phys.Proc.Suppl. **140** (2005) 461, hep-lat/0409116, and hep-lat/0510113
- [18] A. Ceccucci, Z. Ligeti and Y. Sakai, J.Phys. **G 33** (2006) 138
- [19] M. Bona et al., [UTfit collaboration], JHEP **10** (2006) 081; hep-ph/0501199